

MAPPING A KNOT BY A CONTINUOUS MAP

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ABSTRACT. By a fixed continuous map from a 3-space to itself, a knot in the 3-space may be mapped to another knot in the 3-space. We analyze possible knot types of them. Then we map a knot repeatedly by a fixed continuous map and analyze possible infinite sequences of knot types.

1. INTRODUCTION

Let \mathbb{R}^3 be the 3-dimensional Euclidean space. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuous map. Let k be an oriented knot in \mathbb{R}^3 . Then k is a subset of \mathbb{R}^3 that is homeomorphic to a circle. Suppose that the restriction map $f|_k : k \rightarrow \mathbb{R}^3$ is injective. Then the image $f(k)$ is again an oriented knot in \mathbb{R}^3 . In this paper we consider the problem whether or not the knot types of k and $f(k)$ can be chosen arbitrary. We also consider the knot types of the infinite sequence of oriented knots $k, f(k), f^2(k), \dots$.

A knot in \mathbb{R}^3 is usually mapped by an orientation preserving self-homeomorphism of \mathbb{R}^3 . Mapping a knot by a non-injective continuous map is unusual and seems to be a mismatch. We are simply interested in seeing what happens when we relax the condition of homeomorphism to continuous map. Most of the results in this paper states the existence of various examples. After all we see that almost everything can happen when we map a knot by a continuous map. We hope there will be further studies and/or applications of our study.

2. FOLDING A KNOT

We begin with the following imaginary stories. Suppose that there is a right circle drawn on a transparent paper. Fold the paper along a straight line intersecting the circle at two points but missing the center of the circle. Then we find a shape which is no more a right circle but yet a topological circle. Next suppose that a person living in a 4-dimensional space draw a knot on a transparent paper which is of course 3-dimensional, and fold the paper along a flat plane. Then he/she find a new knot. Here we have a question what are the knot types before and after folding the paper. We formulate and answer the question as follows.

We denote the set of all oriented tame knot types in the 3-dimensional Euclidean space \mathbb{R}^3 by \mathcal{K} . Then an element K of \mathcal{K} is an oriented knot type in \mathbb{R}^3 . Note that K is an ambient isotopy class of oriented knots in \mathbb{R}^3 . Therefore an element k of K is an oriented knot in \mathbb{R}^3 . Since K is a tame knot type there is an element k of K that is smooth or polygonal. Here a knot in \mathbb{R}^3 is *polygonal* if it is a union of finitely many straight line segments. All oriented knots in this paper are smooth

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or polygonal unless otherwise stated. Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a map defined by $F(x, y, z) = (x, y, |z|)$.

Theorem 2.1. *Let K_1 and K_2 be elements of \mathcal{K} . Then there is an element k_1 of K_1 such that F maps k_1 homeomorphically onto an element $F(k_1)$ of K_2 .*

Proof. Let D_i be an oriented knot diagram representing K_i and \mathcal{C}_i a set of crossings of D_i such that changing over/under at all crossings in \mathcal{C}_i will turn D_i into a trivial knot diagram D'_i for $i = 1, 2$. Let \mathcal{C}'_i be a set of crossings of D'_i corresponding to \mathcal{C}_i for $i = 1, 2$. Let D be a knot diagram obtained by a diagram-connected sum of D_1 and D'_2 . Let k be an oriented knot whose diagram is D . Then k is an element of K_1 . Suppose that k is slightly above the xy -plane except the arcs corresponding to the under arcs of the crossings in \mathcal{C}_1 and \mathcal{C}'_2 so that the diagram of $F(k)$ is obtained from D by changing over/under at each crossing in \mathcal{C}_1 and \mathcal{C}'_2 . See Figure 2.1. Then the diagram of $F(k)$ is a diagram-connected sum of D'_1 and D_2 . Therefore $F(k)$ is an element of K_2 as desired. See for example Figure 2.2. \square

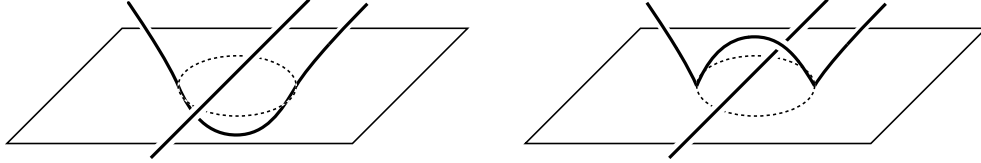


FIGURE 2.1.

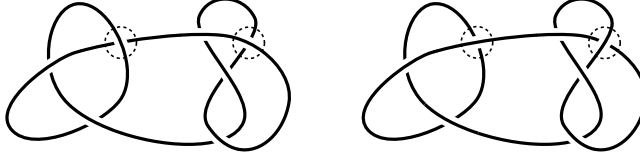


FIGURE 2.2.

3. GENERALIZATION

We now generalize Theorem 2.1 as follows. By a *simple arc* (resp. *disk*, *3-ball*) we mean a topological space homeomorphic to a closed interval (resp. closed 2-disk, closed 3-ball).

Theorem 3.1. *Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuous map. Suppose that there exist a simple arc A and a 3-ball B in \mathbb{R}^3 with the following properties.*

- (1) $A \cap B = \partial A \cap \partial B = \partial A$,
- (2) $A \cup A'$ is a trivial knot where A' is a simple arc in ∂B with $\partial A' = \partial A$,
- (3) f maps each of A and B homeomorphically onto its image,
- (4) there is a simple arc A'' in $\text{int} A$ such that $f(A) \cap f(B) = f(\partial A) \cup f(A'')$ and $(f(B), f(A''))$ is a trivial ball-arc pair,

(5) there are mutually disjoint disks D_1 and D_2 in $\mathbb{R}^3 \setminus f(\text{int}B)$ such that $(D_1 \cup D_2) \cap f(B) = (\partial D_1 \cup \partial D_2) \cap f(\partial B)$ is a disjoint union of two simple arcs and $(\partial D_1 \cup \partial D_2) \setminus f(B) = f(A) \setminus f(B)$.

Let K_1 and K_2 be elements of \mathcal{K} . Then there is an element k_1 of K_1 such that f maps k_1 homeomorphically onto an element $f(k_1)$ of K_2 .

We use the following lemma for the proof of Theorem 3.1. This lemma is a version of Terasaka-Suzuki lemma. A first version of this lemma appeared in an expository paper [5] written in Japanese. Then it is shown in [3, Lemma 1]. See also [6, Lemma 2.1] and [4, Lemma 2.1(1)]. For knot types K_1 and K_2 , $d_G(K_1, K_2)$ denotes the Gordian distance between them. Namely $d_G(K_1, K_2)$ is the minimal number of crossing changes that is needed to deform K_1 to K_2 .

Lemma 3.2. *Let K_1 and K_2 be elements of \mathcal{K} . Then there is an element k_1 of K_1 and an element k_2 of K_2 with the following properties.*

- (1) *There is a 3-ball C in \mathbb{R}^3 such that $C \cap k_1 = C \cap k_2$ and the pair $(C, C \cap k_1)$ is a $(d_G(K_1, K_2) + 1)$ -string tangle,*
- (2) *Outside of C is exactly as illustrated in Figure 3.1.*

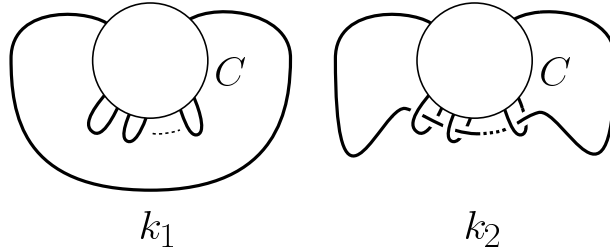


FIGURE 3.1.

Proof. We consider a sequence of $d_G(K_2, K_1) = d_G(K_1, K_2)$ times crossing changes from K_2 to K_1 . We replace a crossing change by a band sum of a Hopf link as illustrated in Figure 3.2. Repeating the replacements we have an element k_2 of K_2 that is a band-sum of $d_G(K_1, K_2)$ Hopf links and an element k_1 of K_1 . See [4, Lemma 2.1(1)] for more detail. By using the deformation illustrated in Figure 3.3 if necessary we can deform the band-sum without changing the knot type so that all Hopf links are contained in a 3-ball, say E , where they are exactly as illustrated in Figure 3.4 (a). After deforming it as illustrated in Figure 3.4 (b) by an ambient isotopy, we see that it is equal to the outside of C of k_2 in Figure 3.1 in the 3-sphere that is the one point compactification of \mathbb{R}^3 . Therefore the outside of E in the 3-sphere give rise to the desired $(d_G(K_1, K_2) + 1)$ -string tangle. \square

Proof of Theorem 3.1. First suppose that the restriction map $f|_B$ of f on B preserves the orientation. We use k_2 in Lemma 3.2. Let G be a 3-ball in \mathbb{R}^3 containing C as illustrated in Figure 3.5. Let α be the component of $k_2 \cap G$ that is disjoint from C . By the assumption there exists an orientation preserving self-homomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $h(G \cup k_2) = f(A \cup B)$ and $h(\alpha) = f(A'')$. Let $\beta = B \cap f^{-1}(h(G \cap k_2) \setminus \alpha)$ and $k_1 = A \cup \beta$. Then we see that k_1 is an element

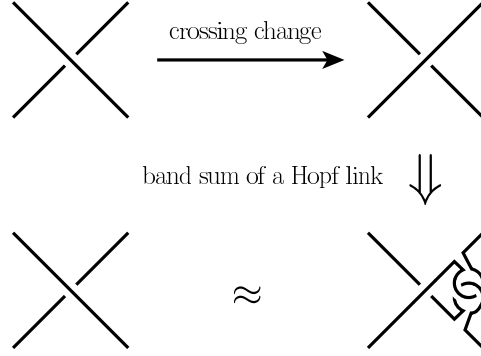


FIGURE 3.2.

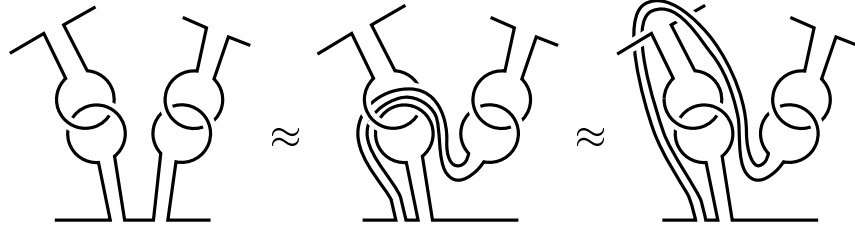


FIGURE 3.3.

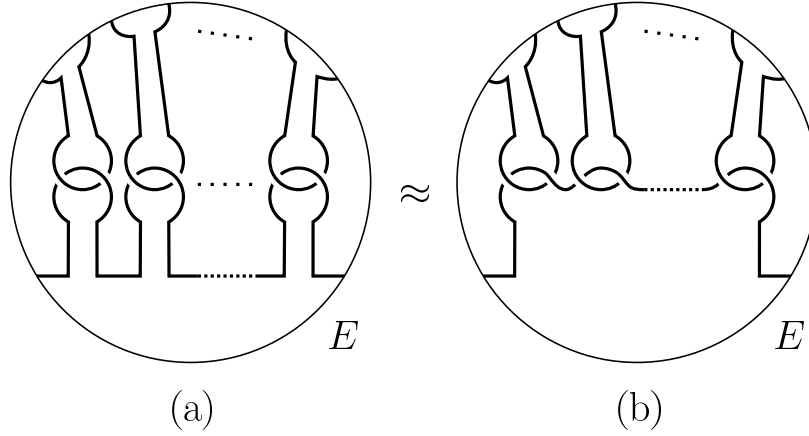


FIGURE 3.4.

of K_1 and $f(k_1) = h(k_2)$ is an element of K_2 as desired. Next suppose that $f|_B$ reverses the orientation. In this case we apply Lemma 3.2 for the mirror image of K_1 instead of K_1 . Then the same proof above works. \square

It is easy to see that the map F in section 2 satisfies the conditions of Theorem 3.1. Therefore Theorem 2.1 follows also from Theorem 3.1. An example of k and

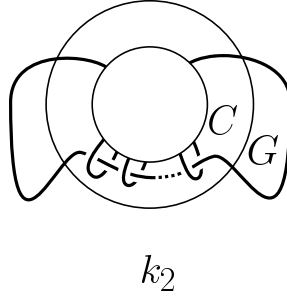


FIGURE 3.5.

$F(k)$ along the proof of Theorem 3.1 is illustrated in Figure 3.6. Here K_1 is the right-handed trefoil knot type and K_2 is the figure eight knot type.

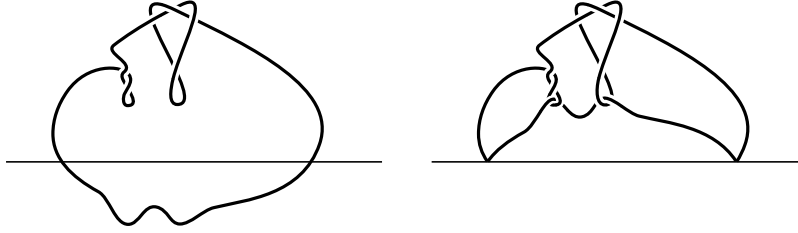


FIGURE 3.6.

As another example of Theorem 3.1 we consider the following continuous map. Let $W : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a map defined by $W(r\cos\theta, r\sin\theta, z) = (r\cos 2\theta, r\sin 2\theta, z)$. It is easy to see that the map W also satisfies the conditions of Theorem 3.1. An example is illustrated in Figure 3.7. Here K_1 is the right-handed trefoil knot type and K_2 is the figure eight knot type.

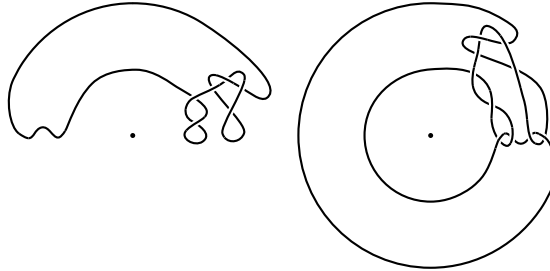


FIGURE 3.7.

Remark 3.3. The point of Theorem 3.1 is that a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is previously fixed. We will show in Proposition 4.2 that for given two oriented knots k_1 and k_2 it is easy to construct a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $f(k_1) = k_2$.

4. MAPPING A KNOT REPEATEDLY BY A CONTINUOUS MAP

Let X be a set and $f : X \rightarrow X$ a map. We define a map $f^n : X \rightarrow X$ inductively by $f^0 = \text{id}_X$, $f^1 = f$ and $f^n = f \circ f^{n-1}$ for each $n \geq 2$. Here $f \circ g$ denotes the composition map of g and f . Let A be a subset of X . We say that A is *iteratively injective with respect to f* if the restriction map $f^n|_A : A \rightarrow X$ is injective for every non-negative integer n . Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuous map. Let k be an oriented knot in \mathbb{R}^3 that is iteratively injective with respect to f . Then $f^n(k)$ is an oriented knot for each n . We analyze the knot types of them. We begin with the following example.

Example 4.1. Let $k = \{((2 + \cos 3\theta)\cos\theta, (2 + \cos 3\theta)\sin\theta, \sin 3\theta) | 0 \leq \theta \leq 2\pi\}$ be a standard $(1, 3)$ -torus knot. Then $W^n(k)$ is a standard $(2^n, 3)$ -torus knot. Thus we have an infinite sequence $k, W(k), W^2(k), \dots$ of knots such that no two of them belong to the same knot type.

Now we state the following proposition for further examples.

Proposition 4.2. *Let k_1 and k_2 be oriented knots in \mathbb{R}^3 . Let V_1 and V_2 be their tubular neighbourhoods in \mathbb{R}^3 respectively. Let B be a 3-ball such that $V_1 \cup V_2$ is contained in $\text{int} B$. Then there exists a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with the following properties.*

- (1) f maps the pair (V_1, k_1) homeomorphically onto the pair (V_2, k_2) ,
- (2) $f(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3 \setminus \text{int} B$ and $f(B) = B$.

Proof. Up to conjugation we may suppose without loss of generality that B is equal to the unit 3-ball \mathbb{B}^3 . Let N_1 be a regular neighbourhood of V_1 contained in $B = \mathbb{B}^3$. Then N_1 is a solid torus and $N_1 \setminus \text{int} V_1$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times [0, 1]$ where \mathbb{S}^1 denotes the unit circle and $[0, 1]$ denotes the unit interval. Let $\varphi : \mathbb{S}^1 \times \mathbb{S}^1 \times [0, 1] \rightarrow N_1 \setminus \text{int} V_1$ be a homeomorphism. Let $g : V_1 \rightarrow V_2$ be a homeomorphism sending k_1 to k_2 . For $\mathbf{x} \in V_1$ we define $f(\mathbf{x}) = g(\mathbf{x})$. For $\mathbf{x} \in \mathbb{R}^3 \setminus \text{int} N_1$ we define $f(\mathbf{x}) = \mathbf{x}$. For $\mathbf{x} \in N_1 \setminus \text{int} V_1$ with $\mathbf{x} = \varphi(a, b, t)$, we define $f(\mathbf{x}) = (1-t)f(\varphi(a, b, 0)) + tf(\varphi(a, b, 1))$. Then we have a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. It is clear that f satisfies the conditions (1) and (2) except the condition $f(\mathbb{B}^3) = \mathbb{B}^3$. Since \mathbb{B}^3 is convex we see that $f(\mathbb{B}^3)$ is a subset of \mathbb{B}^3 . Since f maps $\partial \mathbb{B}^3$ identically onto $\partial \mathbb{B}^3$ and there exist no retraction from \mathbb{B}^3 to $\partial \mathbb{B}^3$ we see that the image $f(\mathbb{B}^3)$ must be the whole \mathbb{B}^3 . \square

Example 4.3. Let V be a knotted solid torus in \mathbb{R}^3 . Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuous map such that the restriction map $f|_V : V \rightarrow \mathbb{R}^3$ is injective, $f(V) \subset \text{int} V$ and $f(V)$ is essential in V . Namely there exists no 3-ball in V containing $f(V)$. Suppose that ∂V and $f(\partial V)$ are not parallel. Let k be a core of V . Then we have an infinite sequence $k, f(k), f^2(k), \dots$ of knots. By a well-known fact on satellite knot we see that no two of them belong to the same knot type. The existence of such f is assured by Proposition 4.2.

Example 4.4. There exists a continuous map from \mathbb{R}^3 to \mathbb{R}^3 that is identical outside a small ball such that a crossing of a knot in the ball is changed by the

map. To be more concrete, we give the following example. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuous map defined as follows.

- (1) $f(x, y, z) = (x, y, z)$ if $|z| \geq 1$ or $\sqrt{x^2 + y^2} \geq 2$,
- (2) $f(x, y, z) = (x, y, 3z + 2)$ if $-1 \leq z \leq 0$ and $\sqrt{x^2 + y^2} \leq 1 + |z|$,
- (3) $f(x, y, z) = (x, y, -z + 2)$ if $0 \leq z \leq 1$ and $\sqrt{x^2 + y^2} \leq 1 + |z|$,
- (4) $f(x, y, z) = (x, y, z - 2\sqrt{x^2 + y^2} + 4)$ if $-1 \leq z \leq 1$ and $\sqrt{x^2 + y^2} \geq 1 + |z|$.

See Figure 4.1 (a) that illustrates f on the xz -plane. Let l_1 be the x -axis $\{(x, 0, 0) | x \in \mathbb{R}\}$ and l_2 the line $\{(0, y, 1) | y \in \mathbb{R}\}$. Then we see that $f(l_1)$ is above $f(l_2) = l_2$ at $(x, y) = (0, 0)$ as illustrated in Figure 4.1 (b).

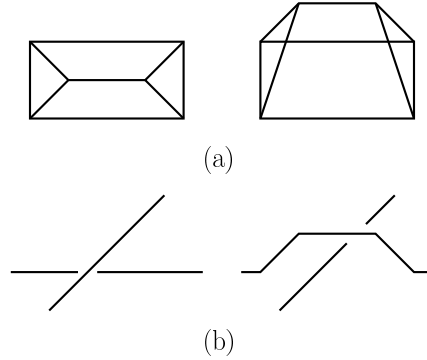


FIGURE 4.1.

Based on Proposition 4.2, we can show the following theorem. We denote the set of all positive integers by \mathbb{N} .

Theorem 4.5. *Let $\varphi : \mathbb{N} \rightarrow \mathcal{K}$ be a map. Then there is a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and an oriented polygonal knot k in \mathbb{R}^3 that is iteratively injective with respect to f such that $f^{n-1}(k) \in \varphi(n)$ for each $n \in \mathbb{N}$.*

Proof. Let B_n be a 3-ball in \mathbb{R}^3 of radius $\frac{1}{3}$ centered at $(n, 0, 0)$ for each $n \in \mathbb{N}$. Let $k_n \in \varphi(n)$ be a polygonal knot contained in the interior of B_n for each $n \in \mathbb{N}$. Let $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a map defined by $g(x, y, z) = (x + 1, y, z)$. Then k_n and $g^{-1}(k_{n+1})$ are oriented knots in a ball B_n . By Proposition 4.2 there exists a continuous map $g_n : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $g_n(k_n) = g^{-1}(k_{n+1})$ and $g_n(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3 \setminus \text{int} B_n$. Let $f_n : B_n \rightarrow B_{n+1}$ be a continuous map defined by $f_n(\mathbf{x}) = g(g_n(\mathbf{x}))$. Then we have $f_n(k_n) = k_{n+1}$ and $f_n(\mathbf{x}) = g(\mathbf{x})$ for each $\mathbf{x} \in \partial(B_n)$. Now we define $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

- (1) $f(\mathbf{x}) = g(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3 \setminus \bigcup_{n \in \mathbb{N}} B_n$,
- (2) $f(\mathbf{x}) = f_n(\mathbf{x})$ for $\mathbf{x} \in B_n$ for each $n \in \mathbb{N}$.

Then we have $f(k_n) = k_{n+1}$ for each $n \in \mathbb{N}$. Let $k = k_1$. Then we have $f^{n-1}(k) = k_n \in \varphi(n)$ for each $n \in \mathbb{N}$ as desired. \square

Let μ be a positive real number. A *tent map* $t_\mu : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$t_\mu(z) = \begin{cases} \mu z & (z \leq \frac{1}{2}) \\ -\mu z + \mu & (\frac{1}{2} \leq z). \end{cases}$$

Tent maps are important examples in discrete dynamical systems theory. See for example [2]. Therefore, as an example of a continuous map from \mathbb{R}^3 to \mathbb{R}^3 that generates a nontrivial discrete dynamical system, we consider a map $\tilde{t}_\mu : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $\tilde{t}_\mu(x, y, z) = (x, y, t_\mu(z))$. More generally we consider the following continuous maps. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous map. Then a continuous map $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $\tilde{f}(x, y, z) = (x, y, f(z))$. We say that a map $f : \mathbb{R} \rightarrow \mathbb{R}$ is *piecewise linear* if \mathbb{R} is a locally finite union of some closed intervals $\mathbb{R} = \bigcup_{\lambda \in \Lambda} I_\lambda$ such that the restriction map $f|_{I_\lambda} : I_\lambda \rightarrow \mathbb{R}$ is an Affine map for each $\lambda \in \Lambda$. By definition a piecewise linear map is continuous.

Proposition 4.6. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise linear map. Let k be an oriented polygonal knot in \mathbb{R}^3 that is iteratively injective with respect to \tilde{f} . Then there exists a finite subset \mathcal{J} of \mathcal{K} such that the knot type of the oriented polygonal knot $(\tilde{f})^n(k)$ is an element of \mathcal{J} for every non-negative integer n .*

For the proof of Proposition 4.6 we prepare the following Proposition 4.7. It extends the well-known fact in knot theory that two knots with the same knot diagram are ambient isotopic. We call a knot diagram without over/under crossing information a *knot projection*. The condition (1) in Proposition 4.7 is a generalization of the condition that two knots have the same knot projection. Then the condition (2) in Proposition 4.7 is a generalization of the condition that two knots have the same over/under crossing information of their common knot projection. Let $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a natural projection defined by $p(x, y, z) = (x, y)$ and let $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a height function defined by $h(x, y, z) = z$. Let k_1 and k_2 be oriented polygonal knots in \mathbb{R}^3 . We say that a homeomorphism $\varphi : k_1 \rightarrow k_2$ is *polygonal* if there exists a subdivisions k_1 such that φ maps each line segment of k_1 to a line segment of k_2 .

Proposition 4.7. *Let k_1 and k_2 be oriented polygonal knots in \mathbb{R}^3 . Suppose that there exists a polygonal homeomorphism $\varphi : k_1 \rightarrow k_2$ that satisfies the following two conditions.*

- (1) $p|_{k_1} = p|_{k_2} \circ \varphi$,
- (2) If $\mathbf{x}, \mathbf{y} \in k_1$, $\mathbf{x} \neq \mathbf{y}$ and $p(\mathbf{x}) = p(\mathbf{y})$, then $(h(\mathbf{x}) - h(\mathbf{y}))(h(\varphi(\mathbf{x})) - h(\varphi(\mathbf{y}))) > 0$.

Then k_1 and k_2 are ambient isotopic.

Proof. For $\mathbf{x} \in k_1$ and $t \in [0, 1]$ we define a point \mathbf{x}_t by $\mathbf{x}_t = (1-t)\mathbf{x} + t(\varphi(\mathbf{x}))$. By the condition (1) we see that the points \mathbf{x} and $\varphi(\mathbf{x})$ are contained in a straight line parallel to the z -axis. Let $j_t = \{\mathbf{x}_t | \mathbf{x} \in k_1\}$. Then by the condition (2) we see that j_t is a polygonal knot in \mathbb{R}^3 . Since $j_0 = k_1$ and $j_1 = k_2$, $\{j_t | t \in [0, 1]\}$ is an isotopy between k_1 to k_2 . Clearly this isotopy has no non-locally flat points. Therefore it is extended to an ambient isotopy. See for example [1, Theorem 3]. \square

It is well-known in knot theory that there are only finitely many knot types that share a common knot projection. We will see in the following proof of Proposition 4.6 that the knots $k, \tilde{f}(k), (\tilde{f})^2(k), \dots$ have the same image under the natural projection $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Note that the multiple points of them are not necessarily only finitely many transversal double points as in the usual knot projection. However it is still true that their knot types can vary only in a finite set as we will see in

the following proof. By $[j]$ we denote the oriented knot type to which the oriented knot j belongs.

Proof of Proposition 4.6. We set $k_n = (\tilde{f})^n(k)$. By the assumption we see that for $m \leq n$ the restriction map $(\tilde{f})^{n-m}|_{k_m} : k_m \rightarrow k_n$ is a polygonal homeomorphism satisfying the condition (1) in Proposition 4.7 and therefore $p(k_m) = p(k_n)$. If the condition (2) in Proposition 4.7 is also satisfied then k_m and k_n are ambient isotopic. We will show that there is a finite set F of non-negative integers such that for every non-negative integer n there exists $m \in F$ with $m \leq n$ so that $(\tilde{f})^{n-m}|_{k_m} : k_m \rightarrow k_n$ satisfies the condition (2) in Proposition 4.7. Then the finite set $\mathcal{J} = \{[k_m] | m \in F\}$ is the desired set.

Note that the set $S = \{(x, y) \in \mathbb{R}^2 | \#(p^{-1}(x, y) \cap k_n) \geq 2\}$ is independent of the choice of a non-negative integer n . Let $k = l_1 \cup \dots \cup l_a$ be the line segments. In the following we pick up some essential cases and explain the idea of proof.

First we consider the case that $p(l_i)$ is a point. Then l_i is a line segment parallel to the z -axis. Then $(\tilde{f})^s(l_i)$ is also a line segment parallel to the z -axis for each non-negative integer s . By the assumption that the restriction map $(\tilde{f})^s|_k : k \rightarrow \mathbb{R}^3$ is injective, $(\tilde{f})^s$ maps l_i homeomorphically onto $(\tilde{f})^s(l_i)$. Therefore $(\tilde{f})^{n-m}$ maps $(\tilde{f})^m(l_i)$ homeomorphically onto $(\tilde{f})^n(l_i)$. For $x, y \in l_i$ with $h(x) > h(y)$, $(h((\tilde{f})^m(x)) - h((\tilde{f})^m(y)))(h((\tilde{f})^n(x)) - h((\tilde{f})^n(y)))$ is positive if and only if both $h((\tilde{f})^m(x)) - h((\tilde{f})^m(y))$ and $h((\tilde{f})^n(x)) - h((\tilde{f})^n(y))$ are positive or both of them are negative. Note that for $z, w \in l_i$ with $h(z) > h(w)$, $h((\tilde{f})^s(z)) - h((\tilde{f})^s(w))$ is positive if and only if $h((\tilde{f})^s(x)) - h((\tilde{f})^s(y))$ is positive. After all, if both $h((\tilde{f})^m(x)) - h((\tilde{f})^m(y))$ and $h((\tilde{f})^n(x)) - h((\tilde{f})^n(y))$ are positive or both of them are negative then $(h(u) - h(v))(h((\tilde{f})^{n-m}(u)) - h((\tilde{f})^{n-m}(v)))$ is positive for any $u, v \in (\tilde{f})^m(l_i)$ with $u \neq v$.

Next we consider the case that there are line segments l_i and l_j with $i < j$ such that both $p(l_i)$ and $p(l_j)$ are line segments and $p(l_i) \cap p(l_j)$ is also a line segment. Note that l_i and l_j may or may not have a common end point. Let x and y be different points in l_i and l_j respectively with $p(x) = p(y)$. Let z and w be another pair of different points in l_i and l_j respectively with $p(z) = p(w)$. Since $(\tilde{f})^s$ maps $l_i \cup l_j$ injectively onto $(\tilde{f})^s(l_i \cup l_j)$ we see that $h((\tilde{f})^s(x)) - h((\tilde{f})^s(y))$ is positive if and only if $h((\tilde{f})^s(z)) - h((\tilde{f})^s(w))$ is positive. Therefore we have the following as in the previous case. If both $h((\tilde{f})^m(x)) - h((\tilde{f})^m(y))$ and $h((\tilde{f})^n(x)) - h((\tilde{f})^n(y))$ are positive or both of them are negative then $(h(u) - h(v))(h((\tilde{f})^{n-m}(u)) - h((\tilde{f})^{n-m}(v)))$ is positive for any $u \in (\tilde{f})^m(l_i)$ and $v \in (\tilde{f})^m(l_j)$ with $u \neq v$ and $p(u) = p(v)$.

As in these two cases we have the following. There are finitely many pairs of points $x_1, y_1, \dots, x_a, y_a$ of k with $p(x_i) = p(y_i)$ for each $i \in \{1, \dots, a\}$ with the following properties. If both $h((\tilde{f})^m(x_i)) - h((\tilde{f})^m(y_i))$ and $h((\tilde{f})^n(x_i)) - h((\tilde{f})^n(y_i))$ are positive or both of them are negative for each $i \in \{1, \dots, a\}$ then $(\tilde{f})^{n-m}|_{k_m} : k_m \rightarrow k_n$ satisfies the condition (2) in Proposition 4.7. Thus we have the desired finite subset \mathcal{J} of \mathcal{K} with at most 2^a elements. \square

We say that a continuous map $f : \mathbb{R} \rightarrow \mathbb{R}$ is *switching* if it satisfies the following condition (*).

(*) For any map $\psi : \mathbb{N} \rightarrow \{-1, 1\}$ there exists $x, y \in \mathbb{R}$ such that $f^{n-1}(x) \neq f^{n-1}(y)$ for each $n \in \mathbb{N}$ and $\frac{f^{n-1}(x) - f^{n-1}(y)}{|f^{n-1}(x) - f^{n-1}(y)|} = \psi(n)$ for each $n \in \mathbb{N}$.

For a switching continuous map $f : \mathbb{R} \rightarrow \mathbb{R}$ and a map $\psi : \mathbb{N} \rightarrow \{-1, 1\}$, a pair (x, y) of real numbers satisfying the condition (*) above is said to be a *realizing pair of ψ with respect to f* .

Proposition 4.8. *Let μ be a positive real number with $\mu > 2$. Then a tent map $t_\mu : \mathbb{R} \rightarrow \mathbb{R}$ is switching.*

Proof. It is well-known that there is a Cantor set $C_\mu \subset [0, 1]$ with $t_\mu(C_\mu) = C_\mu$ such that for $x \in \mathbb{R}$, $\lim_{n \rightarrow \infty} (t_\mu)^n(x) = -\infty$ if and only if x is not in C_μ . Moreover the following is known. See for example [2]. Let \mathcal{S} be the set of all maps from \mathbb{N} to $\{0, 1\}$. For $x \in C_\mu$ we define $s(x) \in \mathcal{S}$ by

$$s(x)(n) = \begin{cases} 0 & \text{if } (t_\mu)^{n-1}(x) < \frac{1}{2} \\ 1 & \text{if } (t_\mu)^{n-1}(x) > \frac{1}{2}. \end{cases}$$

Then $s : C_\mu \rightarrow \mathcal{S}$ is a bijection. Let $d : \mathcal{S} \rightarrow \mathcal{S}$ be a map defined by $d(u)(n) = u(n+1)$. Then we have the following commutative diagram.

$$\begin{array}{ccc} C_\mu & \xrightarrow{t_\mu|_{C_\mu}} & C_\mu \\ s \downarrow & & \downarrow s \\ \mathcal{S} & \xrightarrow{d} & \mathcal{S} \end{array}$$

Let $\psi : \mathbb{N} \rightarrow \{-1, 1\}$ be a map. Let $u : \mathbb{N} \rightarrow \{0, 1\}$ be a map defined by

$$u(n) = \begin{cases} 0 & \text{if } \psi(n) = -1 \\ 1 & \text{if } \psi(n) = 1 \end{cases}$$

and let $v : \mathbb{N} \rightarrow \{0, 1\}$ be a map defined by

$$v(n) = \begin{cases} 0 & \text{if } \psi(n) = 1 \\ 1 & \text{if } \psi(n) = -1. \end{cases}$$

Let $x = s^{-1}(u)$ and $y = s^{-1}(v)$. Then for each natural number $n \in \mathbb{N}$ we have $(t_\mu)^{n-1}(x) < \frac{1}{2} < (t_\mu)^{n-1}(y)$ if $\psi(n) = -1$ and $(t_\mu)^{n-1}(x) > \frac{1}{2} > (t_\mu)^{n-1}(y)$ if $\psi(n) = 1$. Therefore $\frac{(t_\mu)^{n-1}(x) - (t_\mu)^{n-1}(y)}{|(t_\mu)^{n-1}(x) - (t_\mu)^{n-1}(y)|} = \psi(n)$ for each $n \in \mathbb{N}$. Thus t_μ is switching. \square

Theorem 4.9. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise linear switching map. Let $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuous map defined by $\tilde{f}(x, y, z) = (x, y, f(z))$. Let \mathcal{J} be any non-empty finite set of oriented tame knot types. Let $\varphi : \mathbb{N} \rightarrow \mathcal{J}$ be any map. Then there exists an oriented polygonal knot $k \subset \mathbb{R}^3$ that is iteratively injective with respect to \tilde{f} such that $(\tilde{f})^{n-1}(k) \in \varphi(n)$ for each $n \in \mathbb{N}$.*

Proof. Let $\mathcal{J} = \{K_1, \dots, K_m\}$ be a finite set of oriented knot types and $\varphi : \mathbb{N} \rightarrow \mathcal{J}$ a map. We may suppose without loss of generality that $\varphi(1) = K_1$. In the following we do not distinguish a knot diagram and its underlying projection so long as no confusion occurs. Here an underlying projection of a knot diagram is simply a subset of \mathbb{R}^2 and a knot diagram is an underlying projection together with over/under information at each crossing point. Let D_i be an oriented knot diagram representing K_i and \mathcal{C}_i a set of crossings of D_i such that changing all crossings in \mathcal{C}_i will turn D_i into a trivial knot diagram D'_i for each $i \in \{1, \dots, m\}$. Let \mathcal{C}'_i be a set of crossings of D'_i corresponding to \mathcal{C}_i for each $i \in \{1, \dots, m\}$. Let D be a knot diagram obtained by a diagram-connected sum of m knot diagrams D_1, D'_2, \dots, D'_m . Then D is a diagram representing K_1 . Let \mathcal{C} be the set of all crossings of D . We may suppose that $\mathcal{C}_1, \mathcal{C}'_2, \dots, \mathcal{C}'_m$ are subsets of \mathcal{C} . Let $\mathcal{C}_0 = \mathcal{C} \setminus (\mathcal{C}_1 \cup \mathcal{C}'_2 \cup \dots \cup \mathcal{C}'_m)$.

Let k be an oriented polygonal knot in \mathbb{R}^3 whose diagram is D . Namely $p(k) = D$ where $p(x, y, z) = (x, y)$ together with over/under crossing information. For a crossing $c \in \mathcal{C}$ let o_c and u_c be points in k such that $p(o_c) = p(u_c) = c$ and $h(o_c) > h(u_c)$ where $h(x, y, z) = z$. Let $\psi_0 : \mathbb{N} \rightarrow \{-1, 1\}$ be a constant map defined by $\psi_0(n) = 1$ for each $n \in \mathbb{N}$. For each $i \in \{1, \dots, m\}$ let $\psi_i : \mathbb{N} \rightarrow \{-1, 1\}$ be a map defined by $\psi_i(n) = 1$ if $\varphi(n) = K_i$ and $\psi_i(n) = -1$ if $\varphi(n) \neq K_i$. By deforming k by an ambient isotopy of \mathbb{R}^3 that preserves the (x, y) -coordinates if necessary, we may suppose that k satisfies the following conditions. For each $i \in \{0, 1\}$ and each crossing $c \in \mathcal{C}_i$, the pair $(h(o_c), h(u_c))$ of real numbers is a realizing pair of ψ_i with respect to f , and for each $i \in \{2, \dots, m\}$ and each crossing $c \in \mathcal{C}_i$, the pair $(h(u_c), h(o_c))$ of real numbers is a realizing pair of ψ_i with respect to f . We now check that $(\tilde{f})^{n-1}(k) \in \varphi(n)$ for each $n \in \mathbb{N}$. For each crossing $c \in \mathcal{C}_0$ the condition that $(h(o_c), h(u_c))$ is a realizing pair of ψ_0 with respect to f implies that the over/under crossing information of $(\tilde{f})^{n-1}(k)$ at c is identical to that of k . For each crossing $c \in \mathcal{C}_1$ the condition that $(h(o_c), h(u_c))$ is a realizing pair of ψ_1 with respect to f implies that the over/under crossing information of $(\tilde{f})^{n-1}(k)$ at c is equal to that of k if and only if $\varphi(n) = K_1$. For each $i \in \{2, \dots, m\}$ and each crossing $c \in \mathcal{C}_i$ the condition that $(h(u_c), h(o_c))$ is a realizing pair of ψ_i with respect to f implies that the over/under crossing information of $(\tilde{f})^{n-1}(k)$ at c is equal to that of k if and only if $\varphi(n) \neq K_i$. After all we have that the diagram of the knot $(\tilde{f})^{n-1}(k)$ is a diagram-connected sum of D_j with $\varphi(n) = K_j$ and $m - 1$ trivial knot diagrams D'_i with $i \in \{1, \dots, m\} \setminus \{j\}$. Therefore knot $(\tilde{f})^{n-1}(k)$ is a representative of $\varphi(n)$ as desired. \square

Theorem 4.10. *There exists a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with the following properties. For any map $\varphi : \mathbb{N} \rightarrow \mathcal{K}$ there exists a tame knot $k \subset \mathbb{R}^3$ that is iteratively injective with respect to f such that $f^{n-1}(k)$ is a tame knot in \mathbb{R}^3 and $f^{n-1}(k) \in \varphi(n)$ for each $n \in \mathbb{N}$.*

Remark 4.11. The knot k in the statement of Theorem 4.10 may not be a polygonal knot nor a smooth knot. We do not know whether or not we can strengthen the condition on k to be a polygonal knot or a smooth knot.

Proof of Theorem 4.10. A closed subset l of \mathbb{R}^3 is a *long knot* if it is abstractly homeomorphic to \mathbb{R} . By the one-point compactification of \mathbb{R}^3 we have a circle $l \cup \{\infty\}$ embedded in the 3-sphere $\mathbb{R}^3 \cup \{\infty\}$. Since the oriented knot types in the

3-sphere are in one-to-one correspondence with those in the 3-space we may think about the knot type of $l \cup \{\infty\}$ as an element of \mathcal{K} unless it is not a tame knot. Let $t_3 : \mathbb{R} \rightarrow \mathbb{R}$ be a tent map defined above and $\tilde{t}_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a continuous map defined by $\tilde{t}_3(x, y, z) = (x, y, t_3(z))$. Let X be the x -axis of \mathbb{R}^3 . First we will construct a long knot $l \subset \mathbb{R}^3$ that is contained in a neighbourhood $N = \{(x, y, z) \in \mathbb{R}^3 | y^2 + z^2 \leq 9\}$ of X so that l satisfies the following condition. For each $n \in \mathbb{N}$ the knot type of $(\tilde{t}_3)^{n-1}(l) \cup \{\infty\}$ is equal to $\varphi(n)$. Actually l is obtained from X by countably many connected sum of possibly knotted arcs. Namely, for each $m \in \mathbb{N}$ we take a 3-ball B_m of radius 2 centered at $(5m, 0, 0)$ and replace $X \cap B_m$ by a properly embedded arc $\alpha_m \subset B_m$ with $\partial\alpha_m \subset X$. Here α_1 is knotted as the knot type $\varphi(1)$ and all other α_m with $m \geq 2$ are unknotted. As in the proof of Theorem 4.9 we can arrange each α_m so that $(\tilde{t}_3)^{n-1}(\alpha_m)$ is knotted as the knot type $\varphi(n)$ when $m = n$ and unknotted when $m \neq n$. Note that $(\tilde{t}_3)^{n-1}(\alpha_m)$ is not necessarily contained in B_m but the knot type of it is still well-defined. Namely we may suppose that $(\tilde{t}_3)^{n-1}(\alpha_m)$ is a properly embedded arc in $[5m-2, 5m+2] \times \mathbb{R}^2$ and then its knot type is naturally defined. It is easily checked by definition that the map $(\tilde{t}_3)^* : \mathbb{R}^3 \cup \{\infty\} \rightarrow \mathbb{R}^3 \cup \{\infty\}$ defined by $(\tilde{t}_3)^*(\mathbf{x}) = \tilde{t}_3(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$ and $(\tilde{t}_3)^*(\infty) = \infty$ is continuous at ∞ . Since $\mathbb{R}^3 \cup \{\infty\}$ is homeomorphic to a 3-sphere $(\mathbb{R}^3 \cup \{\infty\}) \setminus \{(0, 0, 4)\}$ is again homeomorphic to the 3-space \mathbb{R}^3 . Let $h : (\mathbb{R}^3 \cup \{\infty\}) \setminus \{(0, 0, 4)\} \rightarrow \mathbb{R}^3$ be a homeomorphism. By definition we see that the point $(0, 0, 4)$ is not in $\tilde{t}_3(\mathbb{R}^3)$. Then we have a continuous map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $f(\mathbf{x}) = h((\tilde{t}_3)^*(h^{-1}(\mathbf{x})))$. Let $k = h(l \cup \{\infty\})$. Then k is an oriented knot in \mathbb{R}^3 . The tameness of each knot $f^{n-1}(k)$ follows from the fact that the knot $(\tilde{t}_3)^{n-1}(l) \cup \{\infty\}$ is locally knotted only in a neighbourhood of $(\tilde{t}_3)^{n-1}(\alpha_n)$. This completes the proof. \square

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